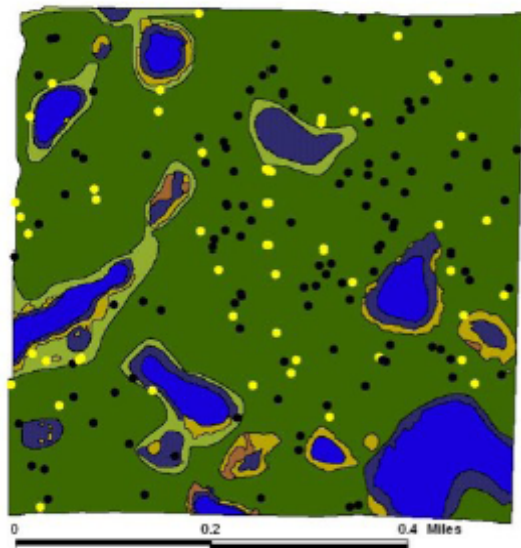


# Spatial Point Patterns

- Observed locations of events: datum is the location
  - Major shift in interest from previous material!
  - Up to now, location has been fixed point or fixed area,
    - Location arbitrary or happenstance, often controlled by the investigator (where to take point samples)
    - Random quantity has been the value at each location
- Random quantity is now the location of an event
- May record additional information at each location  $\Rightarrow$  marked point process
  - Sometimes small # of classes
  - Examples: species of tree, live / dead plant, successful / unsuccessful bird nest, disease case / not diseased person
  - Or, may be continuous quantity
  - Examples: diameter of tree, angle of a crystal
- But that addn info only exists when there is an event at that location

# Duck nests in a 1/4 section of ND



# Potential questions

- ① is intensity ( $\#$  events / unit area) const. or vary. over the study area
  - ② how does intensity vary as function of potential covariates
    - EX: does intensity of duck nests decline with distance to wetland?
  - ③ are events randomly scattered, clustered, or regular
    - EX: are duck nests independently located in space, or do they cluster near other duck nests, or do they avoid being near other nests?
  - ④ how can we describe pattern at multiple scales?
  - ⑤ how can we describe rel. between two (or more) types of points?
    - EX: do depredated nests tend to occur near other depredated nests?
  - ⑥ how can we describe the cor. between marks as a function of distance?
    - when mark is a continuous value
- 
- Historically, 3) was most important Q
  - Now, moving beyond to all the other Q.
  - We'll begin with 3 and 4, then 5 and 6, end w/intensity

# Application areas

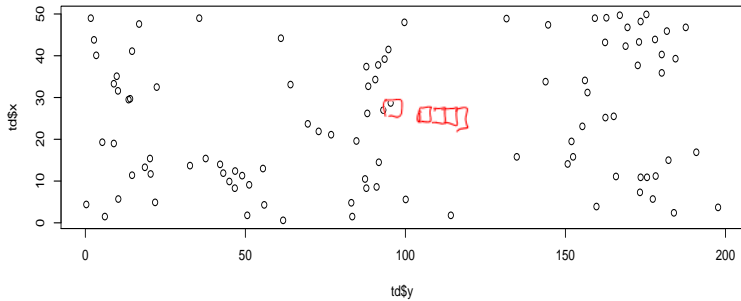
- Many, including:
- Ecology: historically important field of application, many different applications, including:
  - spatial pattern (random / clustered / avoidance) of a single species
  - patterns of mortality (clustered or not?)
- transportation: locations of accidents
- neurology: locations of neurons
- geology: locations of earthquakes (space, or space/time)
- geography: do similar types of stores tend to cluster near each other?
- epidemiology: do cases of a particular disease cluster?
  - If so, suggests contagious disease or single spatial cause

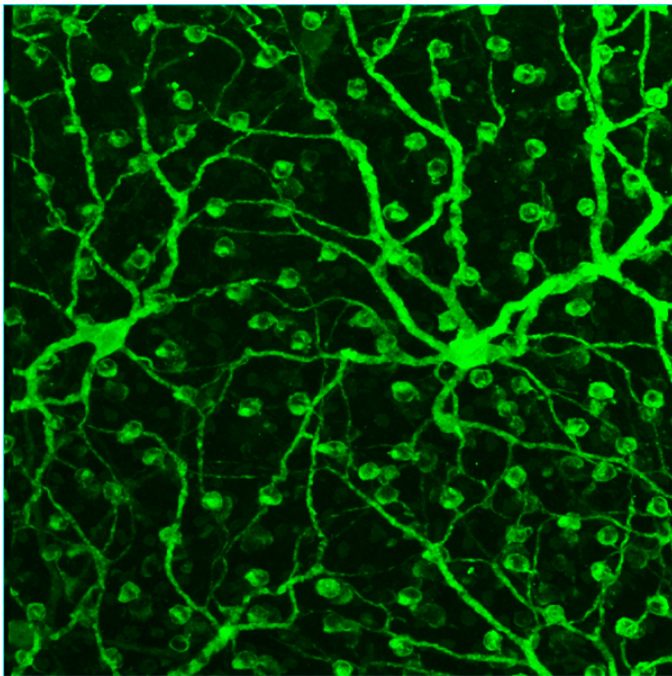
## Examples in pictures:





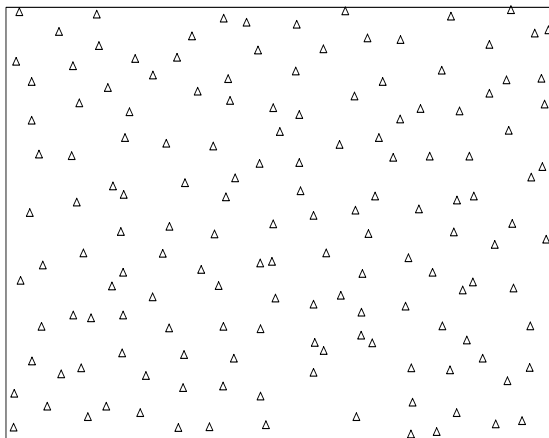
Cypress trees in Good plot 1







Amacrine cells, on



# Homogeneous Poisson Process = Complete Spatial Randomness

- imagine a very small area,  $dA$ , with  $P[\text{event occurs in } dA] = \lambda dA$

- $dA$  small enough that:

- at most 1 event in  $dA$
- most areas have 0 events

- $\lambda = \text{expected \# events} / \text{unit area}$

- $\lambda$  is the intensity of the spatial process

- Two assumptions that give HPP = CSR  $\Rightarrow$  random pattern.

- $\lambda$  constant over study area
- the outcome (0/1) in  $dA_1$  is independent of the outcome in non-overlapping area  $dA_2$

tiny tiny  
area

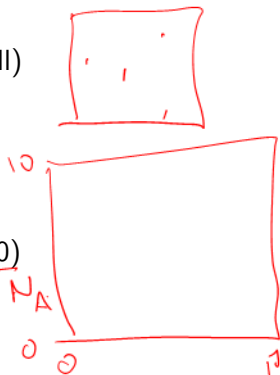
intensity

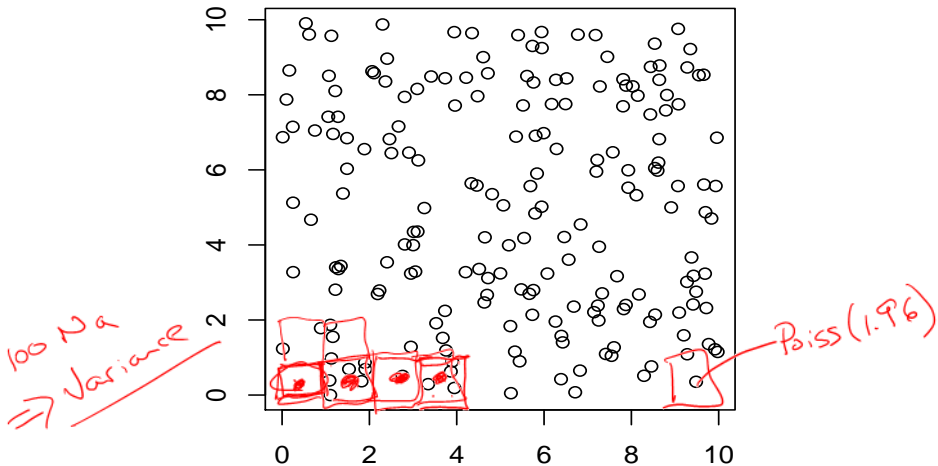
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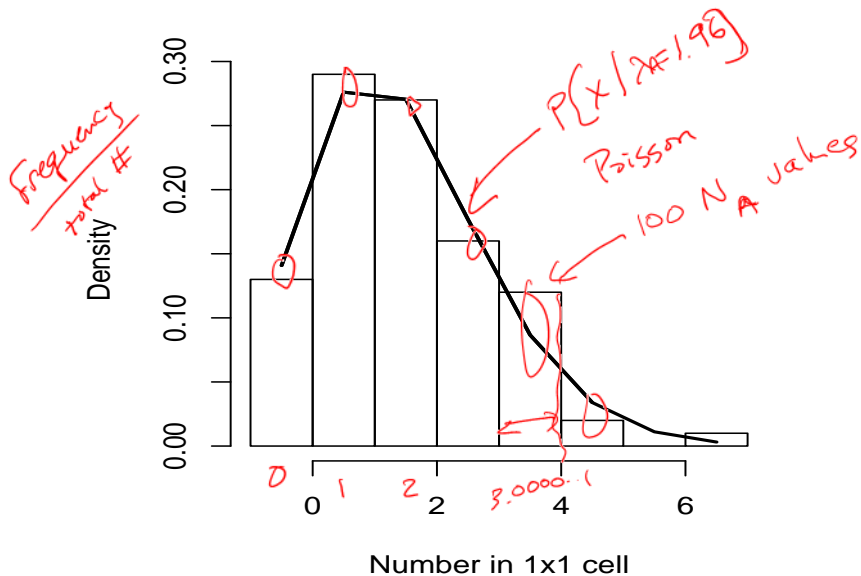
# Homogeneous Poisson Processes

Some mathematical results:

- Define  $N_A$  = # points in area  $A$  (no longer small)
- $N_A \sim \text{Poiss}(\lambda A)$  *when HPP*
  - mean #:  $\lambda A$  *mean  $\lambda$  · area*
  - var #:  $\lambda A$
  - pmf  $P[X | \lambda A] = \frac{e^{-\lambda A} (\lambda A)^X}{X!}$  *mean*
- examples: CSR, observe 196 obs on  $(0,10)$ ,  $(0,10)$
- Look at individual 1x1 quadrats *⇒ 100 values  $N_A$* 
  - mean count per 1x1 quadrats = 1.96
  - Var count = 1.90 *sampling variation*
  - Histogram close to theoretical pmf







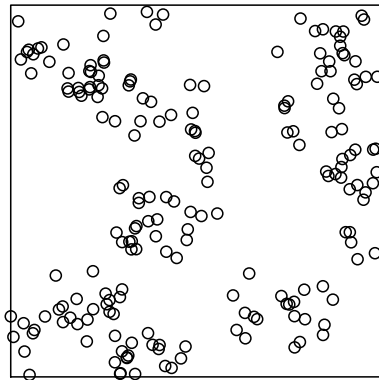
# Clustering

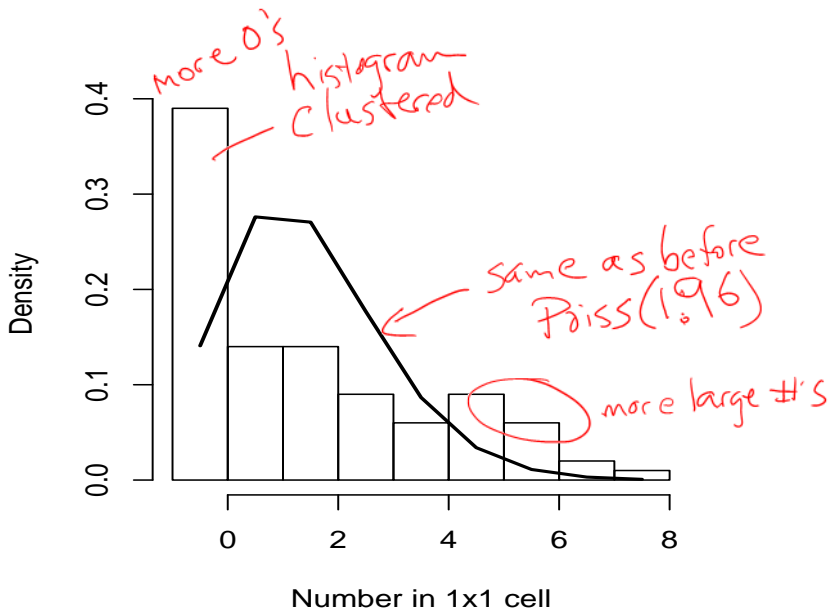


- Clustered processes/pattern: points more likely to occur near other points.
- For quadrats, means that:
  - some quadrats contain a cluster, have more points than expected
  - other quadrats have no points
- same mean, larger variance
- for clustered process with 196 points on next two slides:
  - mean = 1.96, variance = 4.70



# A clustered process

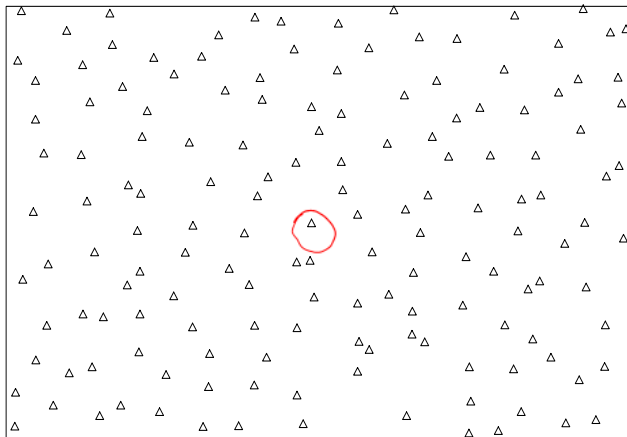








Amacrine cells, on



196 events  
mean 1.96  
variance  
small.  
var mean

# A better approach

- Historically: Quadrats used extensively
- But, very limited. Restricted to one specific scale (size of quadrat)
- Better approach
  - record locations of events, not just count in a box
  - usually all events in a predefined area — *complete enumeration*
  - can be random sample of events —
  - But, hard to take a simple random sample
- Can convert to quadrat counts, but can do a LOT more with (x,y) data

# Summary functions

- Concept:
  - measure something as a function of distance
- Various choices of summary
  - Distance to nearest neighbor (event - event distance)
  - Distance to nearest point (point - event distance)
  - Combination of these two
  - Ripley's K function — last 20 yrs
  - pair correlation function — better
- Each has uses

# Distance to nearest neighbor

- How close is each obs. to its nearest neighbor?

- clustering: NN distances tend to be small
- random (CSR): intermediate
- regular: NN distances tend to be large

$$\hat{G}(x) = \frac{\# \text{ NN} \leq x}{\text{total } \#}$$

- Historical: calculate mean NN distance, compare to theoretical value (Clark-Evans test)

- Current: estimate cdf of NN distance:  $G(x) = P[\text{NN distance} \leq x]$

- for each event: find NN, calculate distance to NN
- hard part is finding NN. Some fancy and fast algorithms (see NN article)
- compare estimated  $\hat{G}(x)$  to theoretical  $G(x)$  for CSR

brute force calculate all distances

	obs 1	obs 2	obs 3	obs 4	obs 5	NN dist = min()
1	0	-	-	-	-	
2	-	0	-	-	-	

# Theoretical CDF on NN distance

CSR

$$P[D < x] = 1 - e^{-\lambda \pi x^2}$$

2D  
area of circle, radius  $x$

- $x$  is distance of concern,
- $\lambda$  is intensity (events per unit area)
- $\pi x^2$  is area of circle, radius  $x$
- (For the statisticians). Nice ex. of CDF method for deriving transforming a random variable
- Define  $D$  = distance to NN dist.

$$\begin{aligned} G(x) &= P[D \leq x] = 1 - P[D > x] \\ &= P[\text{no obs in circle of radius } x] \end{aligned}$$

CSR

$$N_A \sim \text{Poiss}(\lambda A), \text{ so:}$$

$$P[0] = \frac{e^{-\lambda A} (\lambda A)^0}{0!}$$

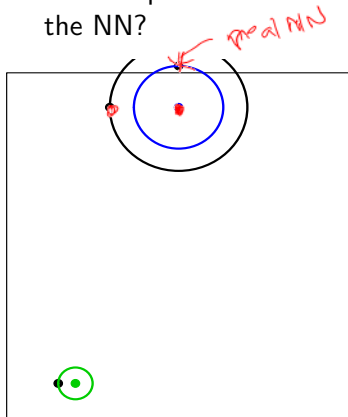
$$P[D \leq x] = 1 - e^{-\lambda \pi x^2}$$

- Similar ideas, different formula for 1D or 3D.



# Edge effects

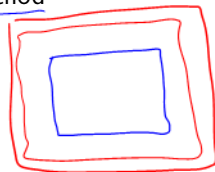
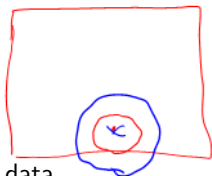
- Above assumes infinite plane
- Real study areas have edges
- When a point is close to edge of mapped area, what is the distance to the NN?



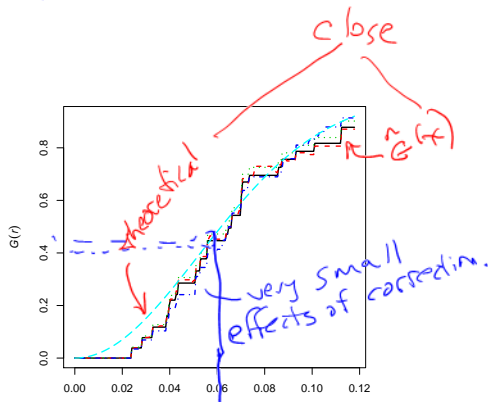
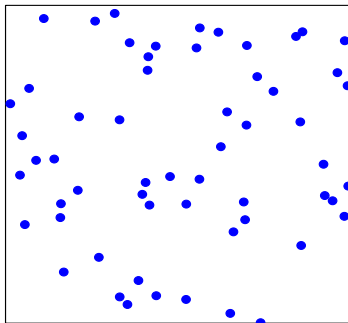
- overestimate  $D$ .
- true NN may be just over the boundary (close to event)
- observed NN (inside study area) is larger than it "should be"
- So underestimate  $G(x)$  especially for large  $x$

# Edge effects

- Three approaches to edge effects
- 1) Ignore problem.
  - Study area edge is a real edge (e.g. lake shoreline)
  - really care about distance to nearest valid event
- 2) Traditional: adjust estimator
  - “edge-corrected” estimator of  $\hat{G}(x)$
  - Usual: use the Kaplan-Meier estimator for censored data
  - others have been proposed, avoid reduced sample method
  - but, bias correction increases Var  $\hat{G}(x)$
- 3) Radical: adjust expectation
  - Use uncorrected estimator
  - Change theoretical  $G(x)$  to account for edge effects
- If goal is to est.  $G(x)$ , 2) much better
- If goal is to test CSR (or some other process), 3) has higher power
- Estimate theoretical  $G(x)$  by simulation







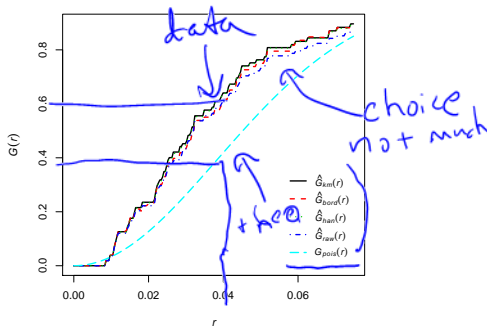
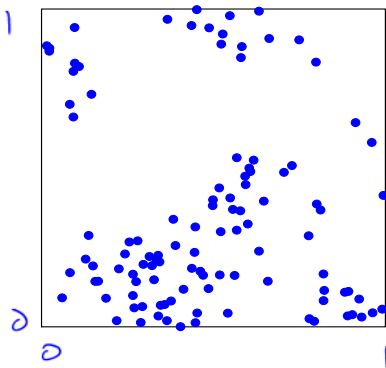
point patterns  
cluster / regular / random = CSR KM method.

---

Summary function  
 $G(x) \approx P[NN \leq x]$

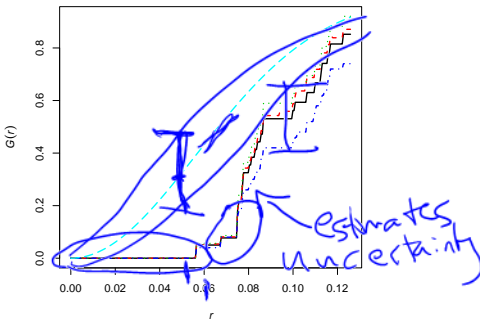
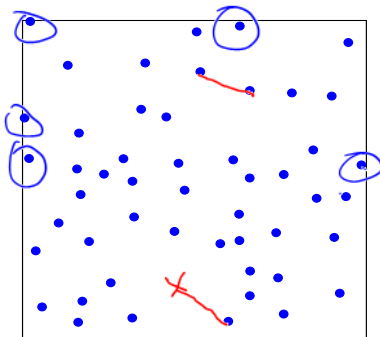
# clustering

- More points close: shorter NN distance



# inhibition / regularity

- Fewer points close: longer NN distance
- Hard core process: no points within a minimum distance



# Sampling variance of $\hat{G}(x)$

- How to calculate  $\text{Var } \hat{G}(x)$ ?
- Quite a hard problem: 1) edge effects
- 2) "Reflexive NN's": pair of points

- B is A's NN, A is B's NN
- same NN distance
- surprisingly common:  $P[\text{reflexive}] = 0.63$  for CSR
- increases  $\text{Var } \hat{G}(x)$  —



- $\text{Var } \hat{G}(x)$  has been derived under CSR, ignoring edge effects

- Now, almost always computed by simulation

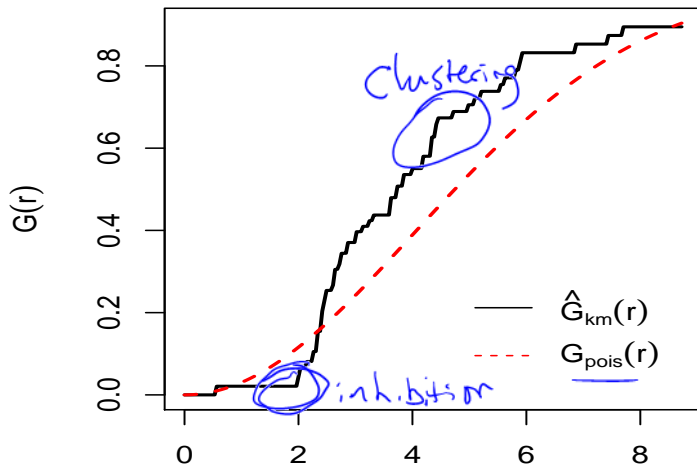
- Simulate a realization of null hypothesis process (e.g. CSR)
- Estimate  $\hat{G}(x)$
- Repeat simulate/estimate 99 or 999 times
- Calculate  $\text{Var } G(x)$  at various  $x$  —

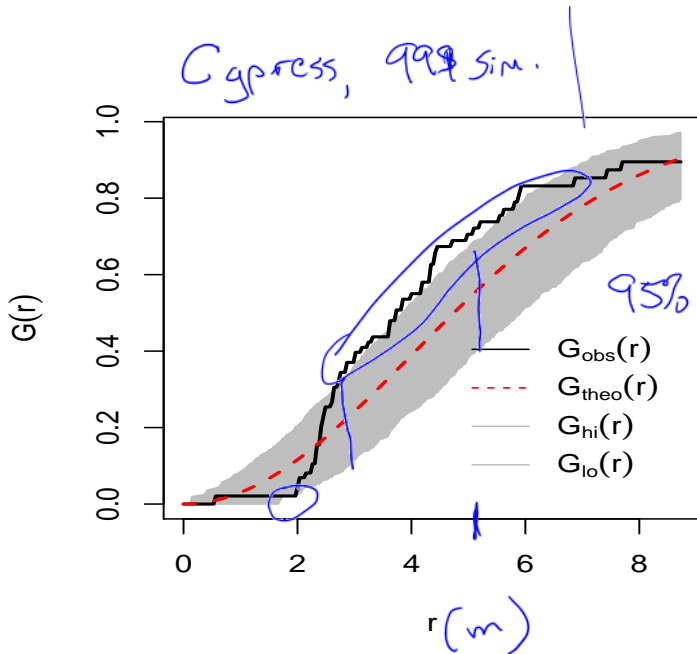
$\text{Var } \hat{G}$   
 $\text{Var } \hat{G}$  given CSR  
 $n_{\text{sim}} + 1$

- Or go straight to a confidence interval

- Calculate 0.025 and 0.975 quantiles of  $\hat{G}(x)$  at a specified  $x$  value
- repeat for various  $x$ 's

# Cypress trees





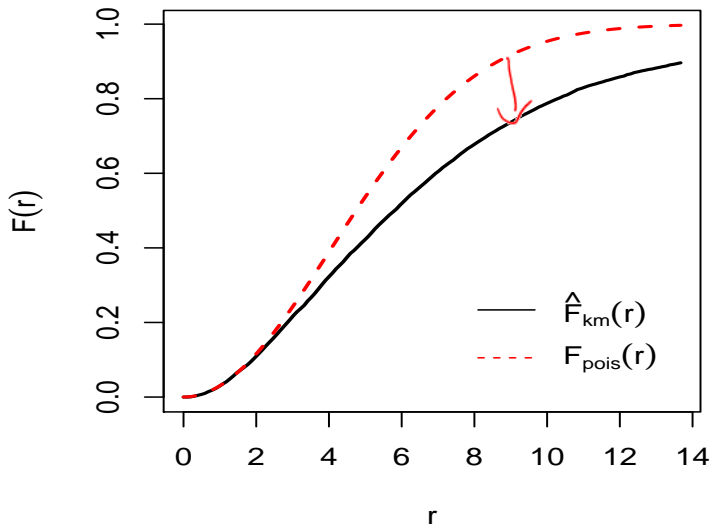
# Point-event distance = empty-space function

$G(x)$ : point - point

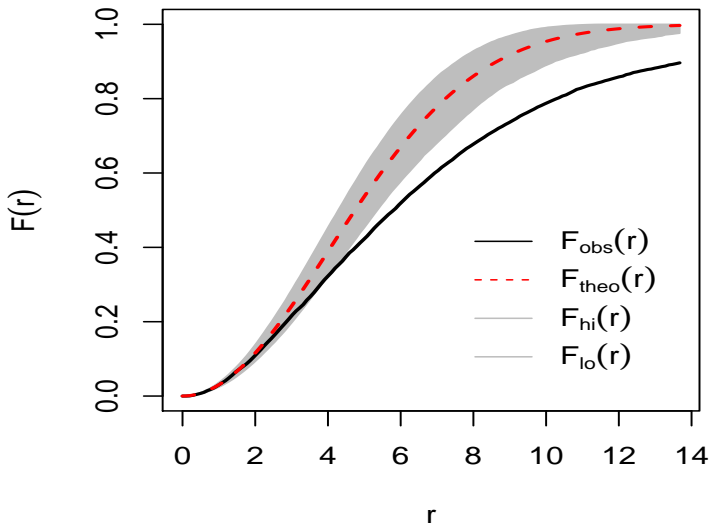
- cdf of distance from randomly chosen point (not an event) to nearest event
- Usually denoted  $F(x)$
- Under CSR, ignoring edge effects:  $F(x) = 1 - e^{-\lambda\pi x^2}$ 
  - same derivation as for  $G(x)$
- But now:
  - large distances  $\Rightarrow$  clustering  
because big areas of empty space
  - small distances  $\Rightarrow$  regularity
- Evaluated in same way as  $G(x)$  ✓
- $F(x)$  more powerful than  $G(x)$  to detect clustering
- $G(x)$  more powerful to detect regularity

$P\{\text{point event} < x\}$







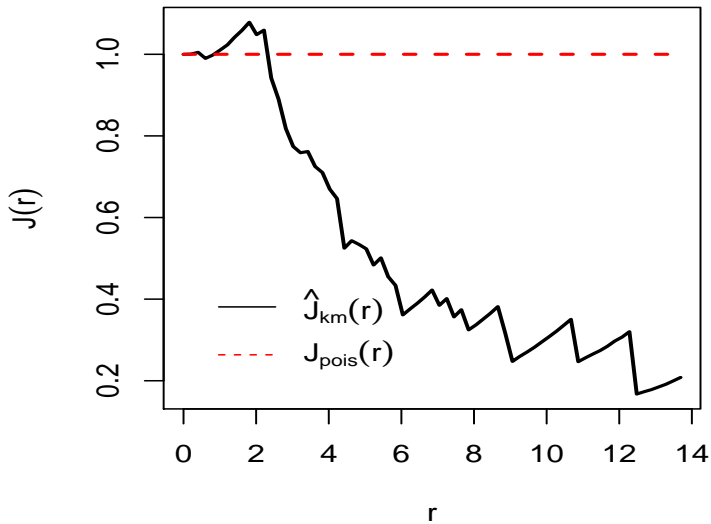


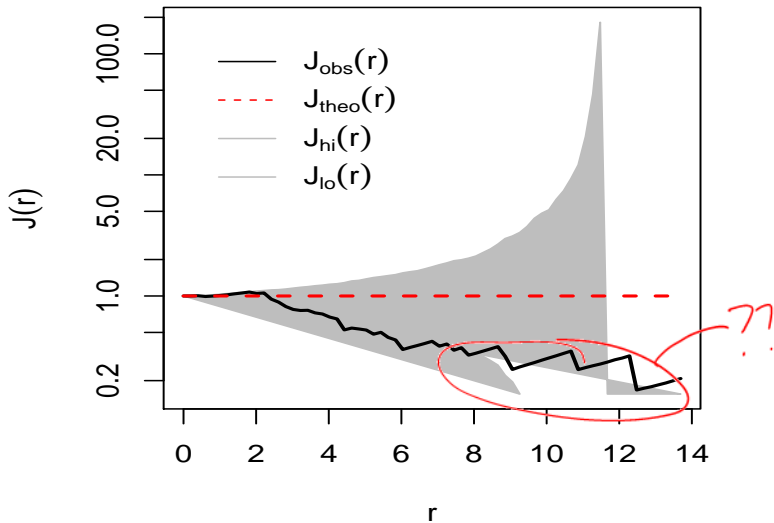
# Baddeley's J function

- Can combine  $F(x)$  and  $G(x)$

$$J(x) = \frac{1 - G(x)}{1 - F(x)}$$

- Interpretation:
  - clustering:  $J(x) < 1$
  - CSR:  $J(x) = 1$
  - regularity:  $J(x) > 1$
- Much newer than  $F(x)$  or  $G(x)$ : 1996 paper
- Few have much experience with it





# Looking at multiple scales simultaneously

$F(x)$  nearest count  
 $G(x)$

- I've emphasized nearest neighbor (of an event, of a point).
- closest event (to the event, to the point)

① • Straightforward extension to 2nd NN (next closest) , 3rd NN, ...

- Gets harder to interpret
- and you have a separate plot for each NN rank

② • Rethink how to compute the summary

- Instead of "how far to closest point"
- think of how many points within a specified distance?

• leads to Ripley's K statistic (Ripley 1976, *J. Appl. Prob*)



do not  
include  
center



contributors  
0

# Ripley's K function

- Like  $F(x)$ ,  $G(x)$ , and  $J(x)$ , looks at 2nd order characteristics of a point pattern
- Now, the most commonly used point pattern analysis function
- Provides information at multiple scales simultaneously

*there*  $K(x) = \frac{1}{\lambda} E(\# \text{ events w/ } x \text{ of an event})$  *add n*

- Interpretation, for each distance  $x$ :

- Clustering:  $K(x)$  large. Many events close to other events at  $x$  or smaller
- Regularity:  $K(x)$  small or 0 at short distances.

- Notes:

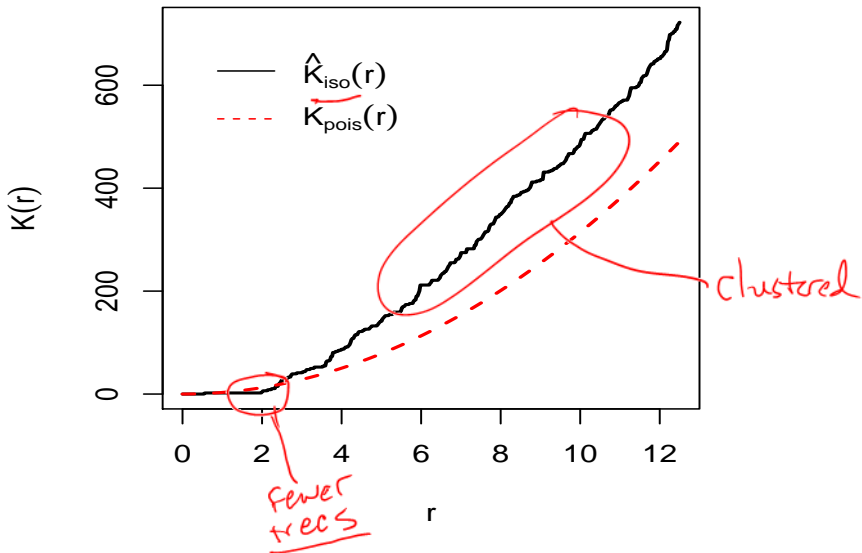
- $K(x)$  can detect clusters of regularly spaced points
- i.e., different patterns at different scales
- but it is cumulative (number of points *within* distance  $x$ )
- we'll see a refinement, the pair correlation function, that looks at points at distance  $x$
- which simplifies (greatly) inferring the scale of a pattern

*obs k: average  
# add 'new'*

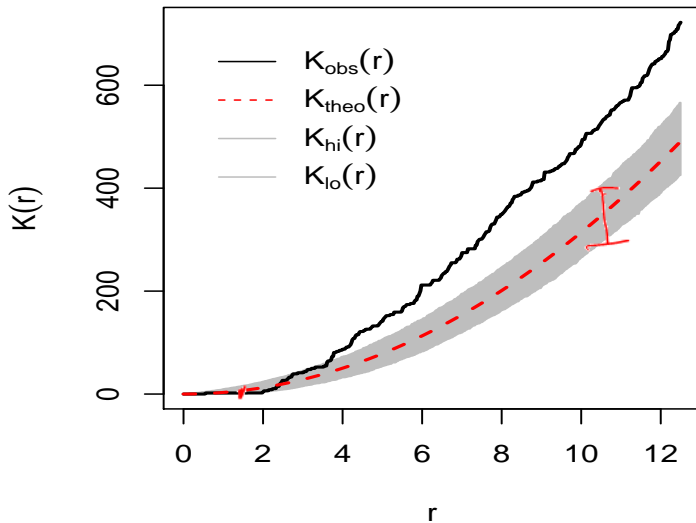
# $K(x)$ under CSR

- Expected value,  $E \hat{K}(x)$ :
  - Under CSR, events are independent,  $E \# \text{ in area } A = \lambda A$
  - $E \hat{K}(x) = E \# \text{ in area } \pi x^2 / \lambda = \lambda \pi x^2 / \lambda = \pi x^2$
- Variance,  $\text{Var } \hat{K}(x)$ :
  - $\text{Var } \# \text{ in area } A = \lambda A$
  - so,  $\text{Var } \hat{K}(x) = \frac{1}{\lambda^2} \text{Var } \# \text{ in } \pi x^2$
  - $= \pi x^2 / \lambda$
  - smaller with more expected points (larger  $\lambda$ )
  - increases with distance,  $x$

any intensity  
CSR  $K = \pi x^2$







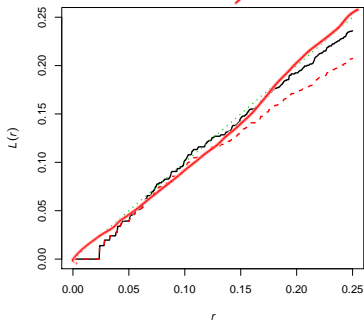
# Besag's L function

- $L(x)$  is a variance equalizing transformation
- if  $Y \sim \text{Pois}(X\beta)$ , then  $\sqrt{Y}$  has constant variance
- Besag's original version

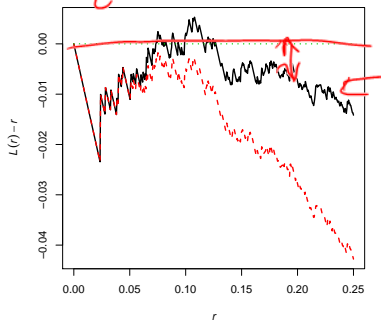
$$\underline{\underline{L(x)}} = \sqrt{\underline{\underline{K(x)}}/\pi} \quad K = \pi x^2$$

- Wiegand and Moloney (2014) call this  $L_1(x)$
- Under CSR:  $L(x) = x$ ,  $\text{Var } \hat{L}(x)$  approx constant.
- I prefer  $L^*(x) = L(x) - x$ 
  - Wiegand and Moloney (2014) call this  $L_2(x)$
- Nice feature of  $L^* = L_2(x)$ : under CSR,  $L^*(x) = 0$
- I believe plots of  $L^*$  are much clearer (but you decide which you prefer)

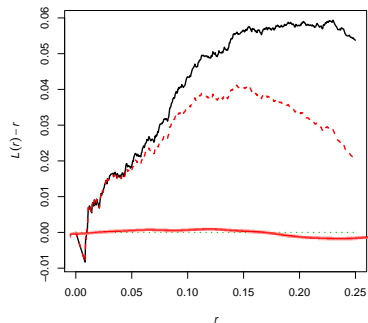
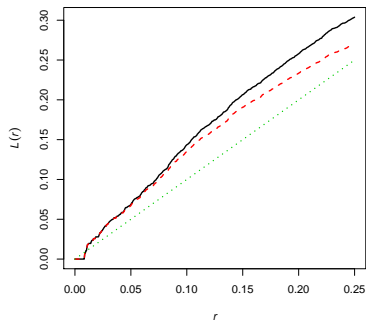
$$L(x) = L_1(x)$$



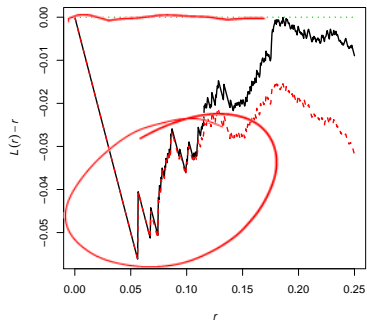
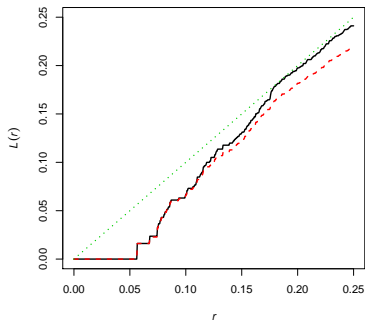
$$L_2(x) = L(x) - x$$

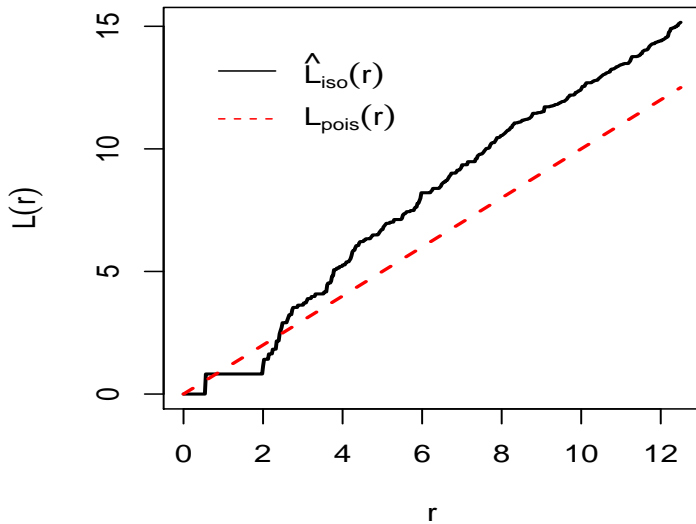


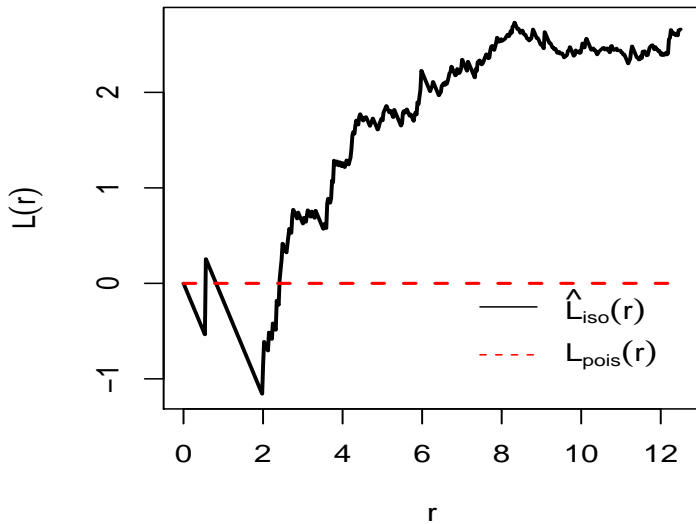
# Clustered



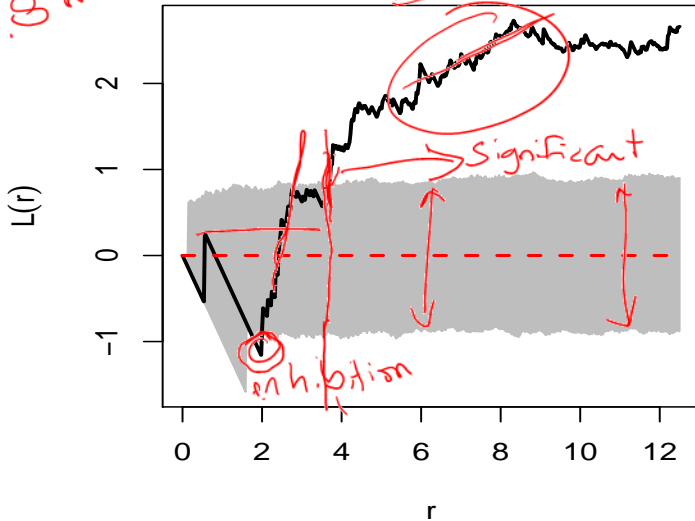
# inhibition







9+ scales  
3.0 m or more





# Pair correlation functions

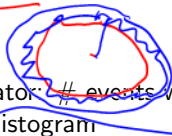
- $K(x)$  and  $L(x)$  are cumulative measures
  - Based on number of events *within*  $x$  of another event
- What if you want to describe association *at* distance  $x$ ?
  - Closer to intuition about spatial scale ==
  - Can untangle multiple processes
    - inhibition at short distances
    - clustering at large distances
- pair-correlation function,  $g(x)$  or  $\rho(x)$

$$g(x) = \frac{1}{2\pi x} \frac{dK(x)}{dx}$$

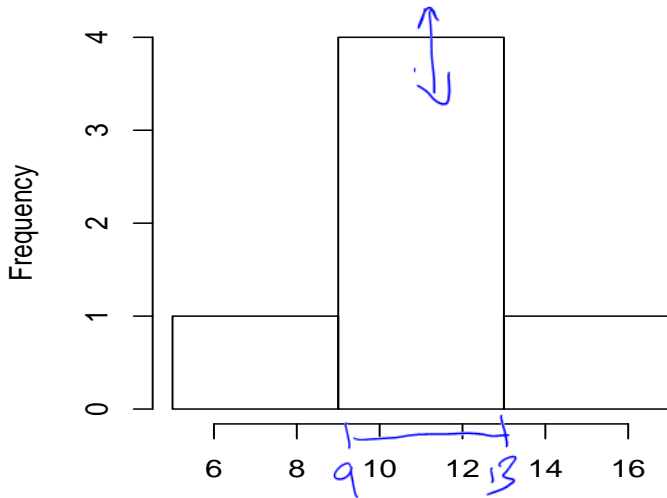
- under CSR ( $K(x) = \pi x^2$ ),  $\frac{dK(x)}{dx} = 2\pi x$ , and  $g(x) = 1$  CSR
  - $g(x) > 1 \Rightarrow$  events more likely AT distance  $x$  than under CSR  $\Rightarrow$  clustering at a scale of  $x$
  - $g(x) < 1 \Rightarrow$  events less likely AT distance  $x$  than under CSR  $\Rightarrow$  repulsion at a scale of  $x$
  - range is  $(0, \infty)$  with 1 as the neutral point
  - so often log transform: evaluate  $\log g(x)$
- $0 \Rightarrow$  CSR  
 $< 0 \Rightarrow$  inhibition  
 $> 0 \Rightarrow$  clustering

# Estimators of $g(x)$

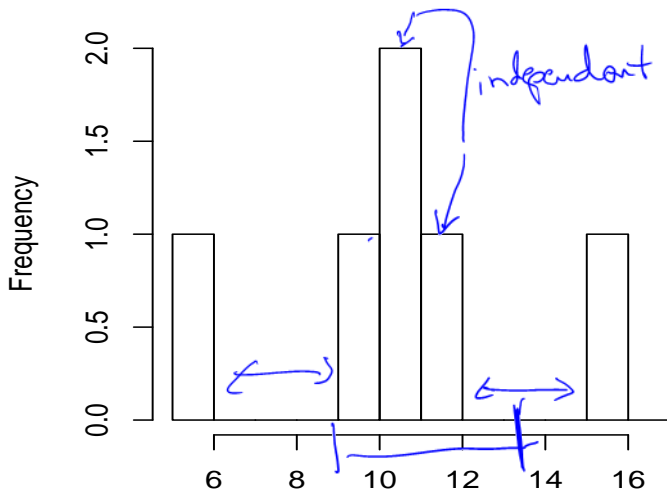
- $g(x)$  is much harder to estimate than  $K(x)$ 
  - $\hat{K}(x)$  is a sum (# events within distance  $< x$ )
  - $\hat{g}(x)$  depends on 0/1 variable: is there an event at distance  $= x$  or not
  - parallel to the issue that a cdf:  $P[X < x]$  is easier to estimate than a pdf:  $f[X = x]$
- Two proposed estimators:
  - Wiegand and Moloney "O-ring" estimator: # events within  $(x, x + dx)$  equivalent to binning obs. to make a histogram
  - ⇒ kernel smoothing: much better (both for density estimation and  $\hat{g}(x)$ )
  - What is the histogram of 5, 10, 11, 11, 12, 16?
    - choice of "bin width" really matters
    - see next two slides



## ~~Histogram~~ with wide categories



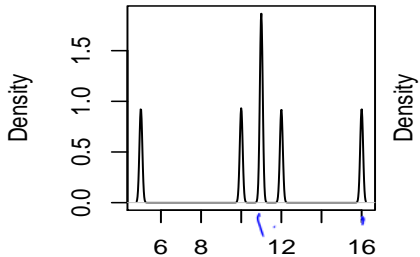
## Histogram with narrow categories



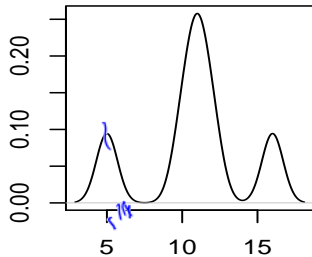
# Kernel smoothing

- Histograms estimate probability density for a range of  $X$  using only the values in that range
- the variance in the estimated probability depends on the number of obs in the bin
  - wide bin: many points, low variance, but biased estimate (one number for many  $X$  values)
  - narrow bin: low bias (small range of  $X$  values), but large variance (few obs in bin)
- Density estimation partially avoids this tradeoff and is less dependent on the breaks between categories
- Concept: superimpose little “bumps” of probability around each obs.  
Add up the probability to estimate  $f(x)$   
*each obs  $\Rightarrow$  many small <sup>bins</sup>*
- result depends on sd of each “bump”
- sd called “bandwidth”
- result also depends on “kernel”, i.e. the shape of the bump
- notice that range of density estimate is wider than data range.

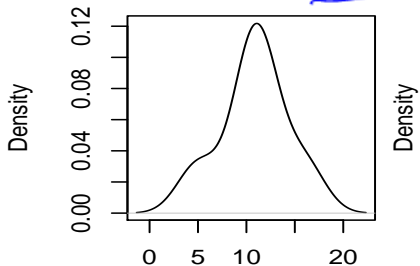




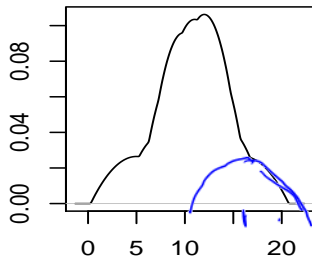
$N = 6$  Bandwidth = 0.07



$N = 6$  Bandwidth = 0.70

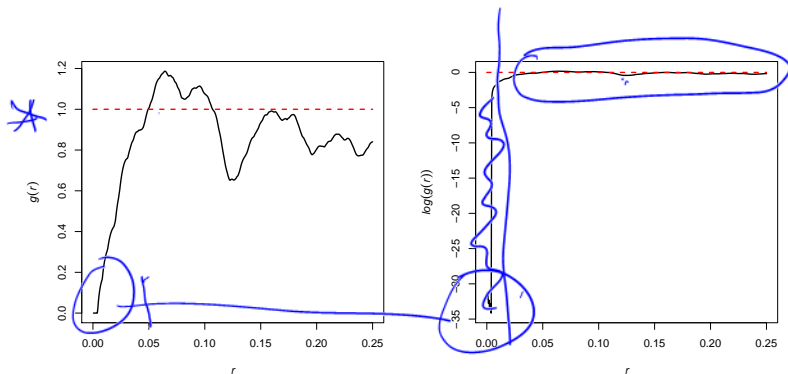


$N = 6$  Bandwidth = 2.11

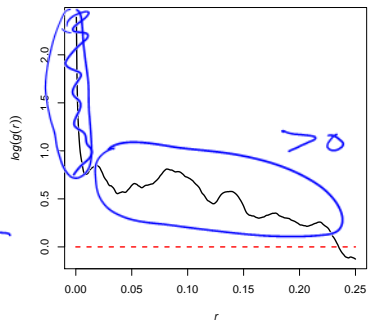
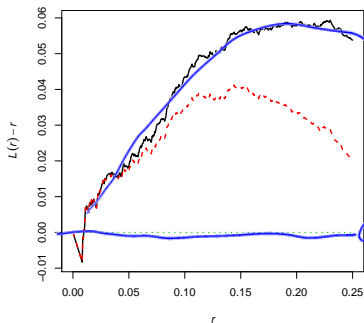


$N = 6$  Bandwidth = 2.1

all pairs of distance

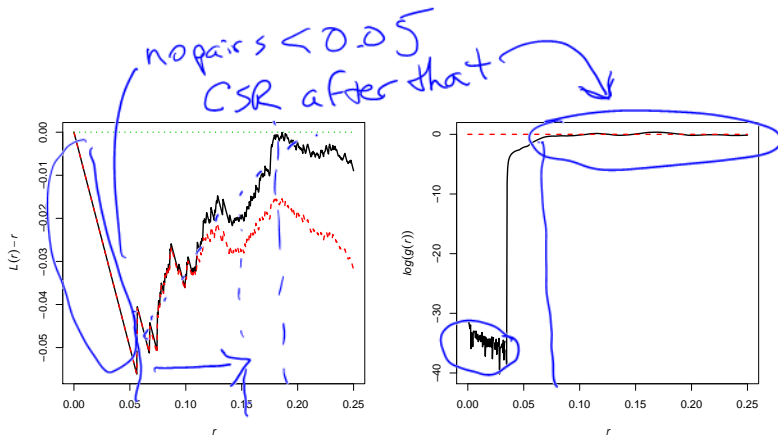


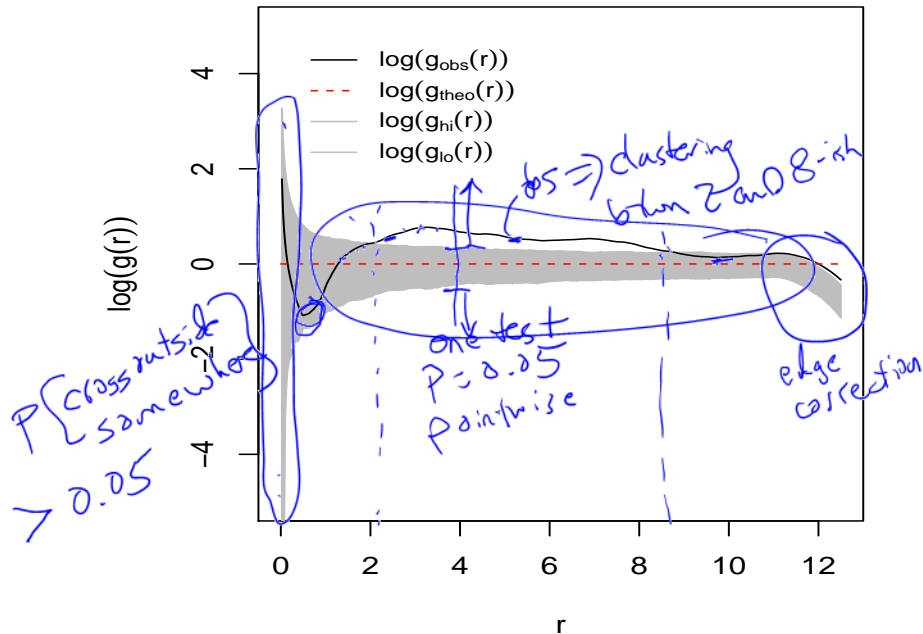
# Clustered





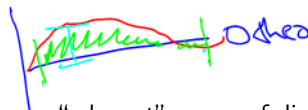
# inhibition





multiple testing.

- 1) to test CSR: pointwise tests
  - estimate  $\hat{L}(x)$  at a range of distances,  $x$
  - use simulation to calculate point-wise quantiles of  $\hat{L}(x)$
  - plot  $\hat{L}(x)$  and simulation envelope
  - interpret deviations above and below expected
  - consider distance  $x_1$ , then distance  $x_2$
  - called pointwise-tests.
  - Type I error rate,  $\alpha$  level, correct for one test
  - One issue (serious): multiple testing
    - doing many tests, one at each distance
    - $P[\text{reject} \text{ --- CSR at any distance}]$  is much larger than  $P[\text{reject} \text{ --- CSR}]$
    - especially for cumulative summary functions,  $K(x)$  and  $L(x)$
    - Quite hard to do a true level  $\alpha$  test
    - usual approaches don't work well because  $\hat{L}(x_1)$  and  $\hat{L}(x_2)$  are correlated

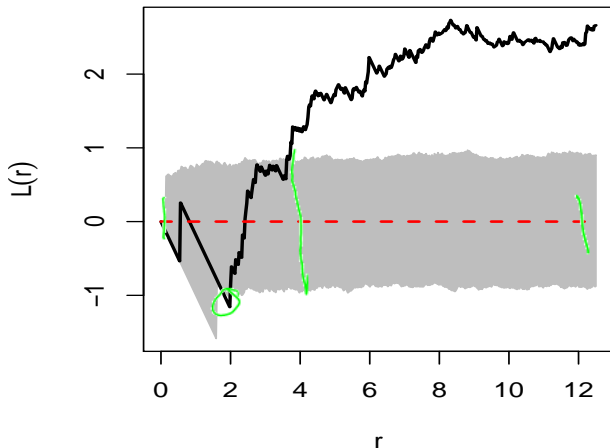


## • 2) summary tests of CSR $L_2$

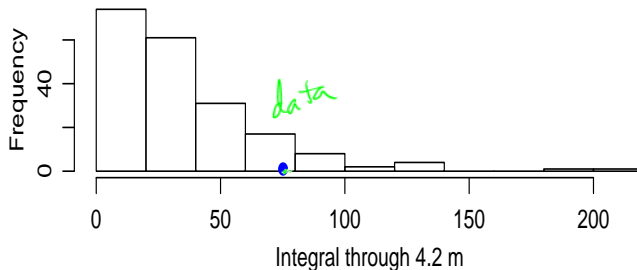
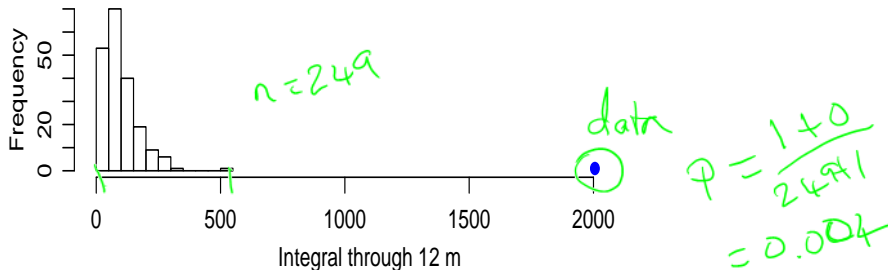
- Calculate a summary statistic across “relevant” range of distances
- Two common choices, using  $L(x)$  as example:
  - $s = \max_x [|\hat{L}(x) - L(x)|]$  (maximum statistic, Maximum Absolute Deviation) *obs theo*
  - $s = \int_x [\hat{L}(x) - L(x)]^2$  (integral statistic, Loosmore and Ford test)
  - Both computed by evaluating “interesting” set of  $x$ , finding max or sum
- $L(x)$  can be theoretical expectation ( $K(x) = \pi x^2$ ,  $L_2(x) = 0$ )
- Or,  $L(x)$  computed as average of  $n$  simulations (see below)
  - accounts for bias due to edge corrections
- Integral better when consistent but small deviations above expected curve
  - more commonly used
- Max better when large excursion from theoretical value for a small range of distances

- Turning into a hypothesis test
  - Have  $s_{obs}$  from the observed pattern
  - Simulate many (39, 99, 999) random patterns under  $H_0$  (e.g., CSR)
  - Calculate summary statistic for observed data and each simulated data set
  - Calculate  $P[\text{as or more extreme summary statistic}] = \text{p-value}$
  - Usually one-sided definition of more extreme (only care about large  $s$ )
- This avoids multiple testing issues and gives valid p-value

# Summary tests: Cypress pointwise tests



# Summary tests: Cypress summary tests

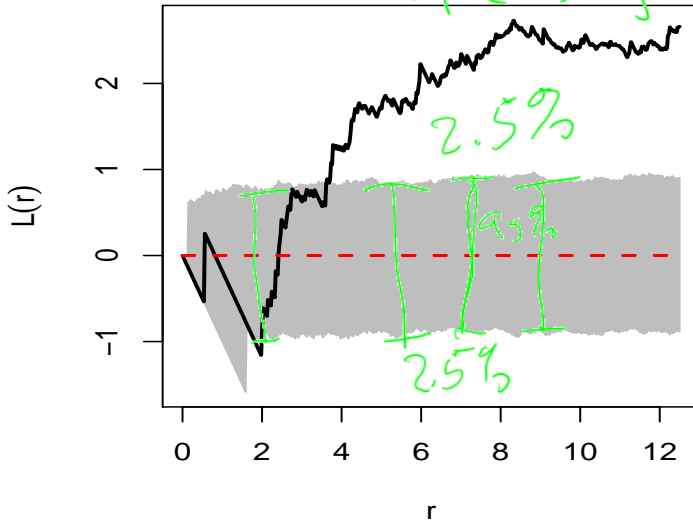


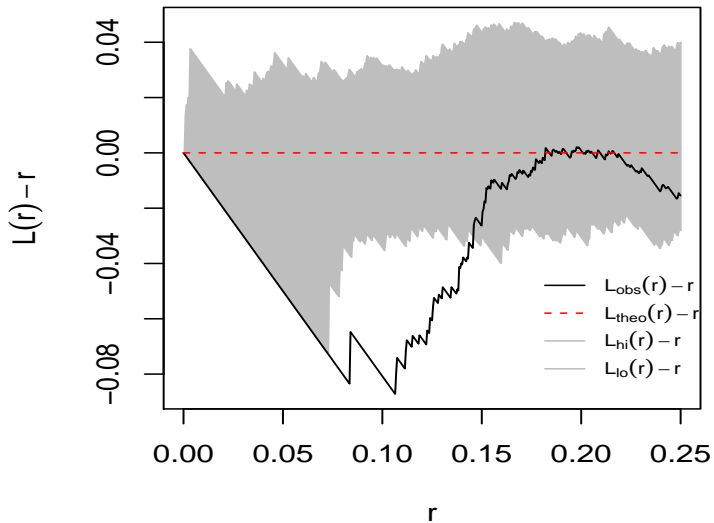
# Summary tests

- Need to choose upper and lower distances
  - best when chosen to be biologically relevant.
  - DON'T look for the most significant region
- most commonly used to test CSR
  - But you specify the null hypothesis
  - same approach can be used for any point process model (examples coming soon)
- The integral and especially the maximum statistic assume  $\text{Var } \hat{L}(x)$  approximately constant
  - Don't use  $K(x)$ : because  $\text{Var } K(x)$  is definitely not constant
  - Use  $L(x)$  instead, approximately constant variance
  - But not perfect (see next two plots)
  - Can transform  $G(x)$  or  $F(x)$  (both proportions), e.g.  $\sin^{-1} \sqrt{\hat{G}(x)}$
- There are studentized summary statistics, if unequal Var is bad

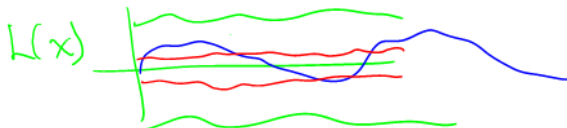


any deviation = Summary  
= multiple testing





# A poor use of a test of CSR



- Until a few years ago, it was fashionable to
  - map locations of all things in an area. Usually trees or other plants, could be animal nests
  - usually many species
  - for each species, test CSR (usually using K/L functions) ✓
  - tabulate # species that are clustered, # random, # regular ✓
- then make ecological conclusions about the community
- do you see the issue here?  $>95\%$   $<95\%$  in 95% level.

no evidence of a diff  $\neq$  no diff  
accept  $H_0$   $H_0$  is true  
depends on sample size

# A poor use of a test of CSR

- If you don't reject  $H_0$ : CSR for a species, do you know that species is randomly distributed?
- What if you only had 10 individuals for that species?
  - Statistical power to detect 'not-CSR' is really small
- failure to reject  $H_0$  does not  $\Rightarrow H_0$  is true
- In my experience (mostly with trees)
  - large # events: detect clustering, sometimes regularity *not random*
  - small # events: accept  $H_0$
- If you expect intensity to vary over a study area, that introduces clustering.
- If you believe that non-random spatial patterns are the norm, the hypothesis test is really telling you only whether you have a sufficiently large sample size to detect that non-random pattern.

## • 3) Estimating # “excess events”

- if pattern is CSR, expect  $\lambda\pi x^2$  events within radius  $x$  of another event
- data says an average of  $\lambda\hat{K}(x)$  events within radius  $x$  of another event
- $\lambda\hat{K}(x) - \lambda\pi x^2$  is average “excess events”
- describes magnitude of clustering in subject-matter terms
- less frequently used is  $\frac{\hat{K}(x)}{\pi x^2} - 1$   $\Rightarrow k = \frac{\hat{K}(x)}{\pi x^2}$   
proportion of excess events at distance  $x$

## • Cypress tree illustration

- $\hat{\lambda} = 98 / (50 \times 200) = 0.0098$
- at distance of 10m,  $\hat{K}(10) = 488.6 \Rightarrow$  ave. of 4.7 cypress trees within 10m of another cypress tree
- $\pi 10^2 = 314.7 \Rightarrow 1.7$  excess cypress trees within 10m of another.
- Or,  $\frac{488.6}{314.7} - 1 = 0.55 \Rightarrow 55\%$  more cypress trees within 10m of another.

- 4) Describing spatial scale  $g(x) = \text{pair correl}$ 
  - “scale” is a tricky concept. Various definitions
  - Here, scale = distance(s) at which events repulse each other or attract each other
  - A distance-specific concept
  - Many studies have used  $\hat{K}(x)$  or  $\hat{L}(x)$  to estimate scale, e.g. find  $x$  where  $L(x)$  is most different from theoretical value
- Increasingly understood to be wrong
  - Both  $\hat{K}(x)$  and  $\hat{L}(x)$  are cumulative functions: # points within circle of radius  $x$
  - Small # at distance  $x$  may be because repulsion (fewer pts.) at distances  $< x$ , even if strong clustering at  $x$
  - Really want to know what is going on AT distance  $x$ , not  $\leq x$
- Use pair-correlation function



- 5) How precise is  $\hat{L}(x)$  or  $\hat{g}(x)$ ?
- Not the width of the Null hypothesis envelopes
- Precision of  $\hat{L}(x)$  or  $\hat{g}(x)$ 
  - Certainly depends on  $N = \#$  points
  - But also on the spatial pattern
    - $\hat{K}(x)$  more variable for clustered patterns
- If you know the true spatial pattern, simulate from that pattern and calculate envelope
- If you don't know the true pattern, use a bootstrap  $\Rightarrow$  data  $\Rightarrow c_i$
- Point pattern bootstrap proposed by Loh, 2008

not CSR  
different variance  
not simulation  
envelope

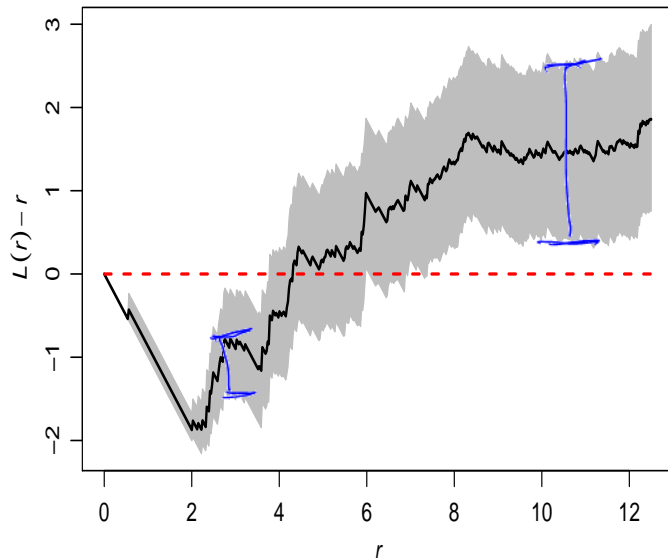
use estimated model  
data  $\Rightarrow c_i$

# The bootstrap

- General method for estimating precision of a statistic
- Uses resampling the data to approximate the unknown sampling distribution of a statistic
- Gives you the se of a statistic or a confidence interval for a statistic
- CI much more common
- Not the same as a randomization test or a null hypothesis test
  - Hypothesis test: simulate / resample assuming  $H_0$  (CSR, no diff. in means)
  - Bootstrap: simulate / resample assuming  $H_a$  (arbitrary pattern, non-zero diff)
- Extremely useful tool for “difficult” problems *boot library*
- Usual forms of bootstrap don't work for point pattern data
- Problem is that one point contributes to many  $L(x)$
- Loh devised something that (so far) is acceptable sometimes
  - resample contributions to  $\hat{g}(x)$  or  $\hat{L}(x)$
- Issues when bootstrap average curve not same as data curve (see below)

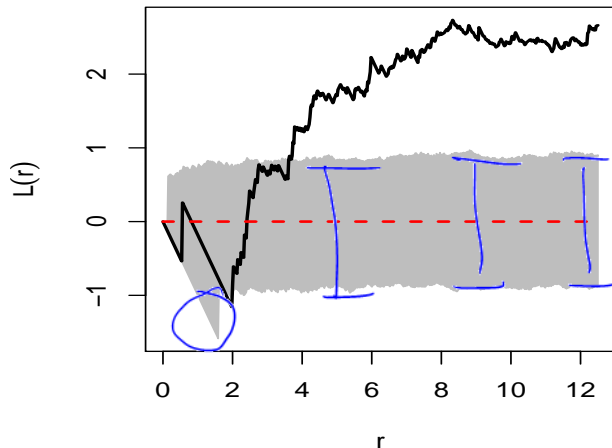


# Cypress L(x) bootstrap

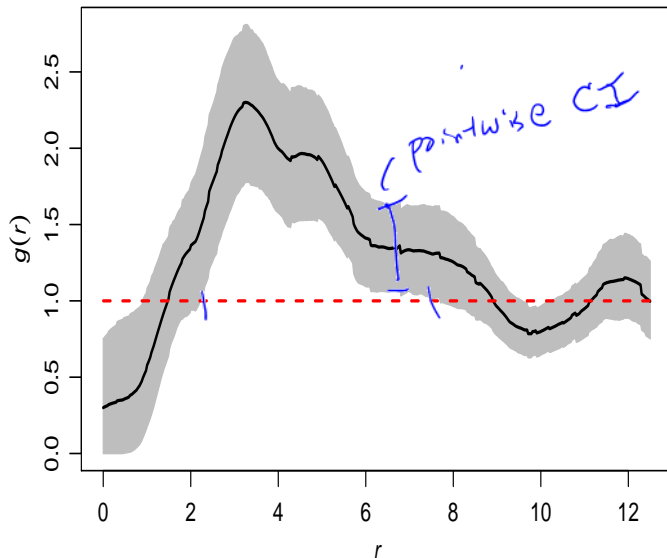


Handwritten blue lines and the text "CSR" are present to the right of the plot.

# Cypress L(x) CSR (null) envelope



# Cypress g(x) bootstrap



Summary function  
descriptive  
evaluating  
clustered  
apparently random  
inhibition  
General issues  
multiple test  
interpretation  
bootstrap  $\Rightarrow$  CI

# Cypress $g(x)$ CSR (null) envelope

